A fully implicit model for simulating dynamo action in a
Cartesian domain

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Abstract

We present a fully implicit numerical method to solve the incompressible MHD equations in a strongly rotating Cartesian domain. The equations are solved in a primitive variable formulation using a finite volume discretization. In order to use massively parallel computers, we applied a domain decomposition approach in space. The performance of this model is compared with an earlier model, which treated the convective terms of the equations in an explicit manner. Our results indicate that although the fully implicit method needs about three times the memory of the implicit–explicit method, it is superior in terms of computational efficiency. As an application of this model, we investigated the influence of the Prandtl number in the range of 0.01–1000 on the dynamics of the dynamo. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Cartesian domain; Dynamo action; Implicit–explicit method

1. Introduction

The dynamo action operating in the molten outer core of the Earth seems today the only vital mechanism able to generate and maintain the Earth’s magnetic field. The underlying idea that a flow in an electrically conducting fluid maintains a magnetic field by electromagnetic induction, which would decay by Ohmic dissipation, otherwise, may appear rather simple. Due to its nonlinear nature, however, the study of stability properties of the magnetohydrodynamic problem, and especially the investigation of finite amplitude properties, poses one of the most fascinating and likewise difficult problems in geophysical fluid dynamics.

Today, there is almost a consensus that the geodynamo operates in the so-called ‘strong-field regime’ (Roberts, 1988). In the ‘weak-field regime’, the dynamo action is characterized by a balance between pressure and Coriolis forces while buoyancy, Lorentz, inertial and frictional forces are understood as small perturbations to the weak-field flow (Busse, 1975). Characteristically, weak field dynamos do generate a magnetic field but the flow is not influenced by the generated field. In strong-field dynamos, the magnetic energy (ME) is, typically, at least one order of magnitude higher than the kinetic energy (KE) and there exists a tight coupling between the magnetic field and the flow where it stems from.
Applying numerical models to the problem of strong-field dynamos has recently led to considerable progress. Glatzmaier and Roberts (1995) were able to demonstrate the existence of a self-consistent dynamo, which does not only generate a magnetic field but also resembles some of the typical features of the magnetic field of the Earth. A different but similar model has been employed by Kuang and Bloxham, (1997) in order to investigate the role of the viscous boundary layers at inner/outer core interface. Christensen et al. (1998) observed some of the characteristic features even at lower values of the rotation rate and pointed out that the dipole-dominated field seems to be a robust feature.

The recent progress has impressively demonstrated that numerical modeling momentarily forms the key instrument for research on dynamo problems of planetary scale. It is also clear that numerical modeling of the dynamo process is at the edge of today’s generation of computers. Glatzmaier and Roberts, 1995 report on numerical experiments, which need thousands of CPU-hours on a Cray-vectorcomputer. Even these demanding models, though doubtless very impressive, resemble the Earth core only roughly. Thus, only the most robust conclusions from dynamo modeling can be regarded as relevant for the geodynamo problem. Questions like “Can inertia be neglected?”, or how to treat the boundary between inner and outer core, remain unanswered up to now.

Most of the mentioned dynamo models are based on a pseudo-spectral formulation (Glatzmaier and Roberts, 1995; Kageyama and Sato, 1997; Kuang and Bloxham, 1997). While spectral methods lead to very efficient computer codes on vector machines, they cannot easily be parallelized. This drawback will become more important in the future when cluster arrangements based on commodity hardware will be available on an unsurpassed price/performance ratio (Warren et al., 1997).

We have, therefore, developed a dynamo model for parallel computers, which is based on a finite volume discretization (Trompert and Hansen, 1999). As explained below, the finite volume discretization is particularly well suited for a parallel approach.

At the present stage, the model allows the investigation of a Childress—Soward dynamo type in Cartesian coordinates (Childress and Soward, 1972). There have already been several models that are able to simulate dynamo action in a Cartesian domain (e.g., Soward, 1974; St. Pierre, 1993; Jones and Roberts, 1999). To our knowledge, all these models employ spectral approaches in the horizontal direction. We have chosen a Cartesian domain in order to be able to reach more realistic parameters (especially lower values of the Ekman number) than the existing spherical codes. The model philosophy is to delineate particular physical phenomena rather than to obtain a dynamo that resembles the geodynamo as closely as possible.

In this paper, we address two main points. At first, we will describe the properties of a fully implicit time stepping method. We will concentrate on the question if the increase in memory requirements and computational effort is met by the larger time step size. Secondly, we report on experiments that have been carried out to investigate the importance of inertia and viscous forces as compared to rotation. Previous work on thermal convection has indicated that the Prandtl number plays an important role with respect to the temporal and spatial behavior of thermal convection. In what follows, we present a study on the role of the Prandtl number on dynamo properties.

2. Model and methods

2.1. Governing equations

The equations describing incompressible magnetohydrodynamics in a 3D Cartesian domain, which rotates around the vertical (\( \hat{z} \)) axes are given in their non-dimensional form as:

\[
\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot \left( \mathbf{u} \mathbf{u} - \frac{Pr}{p} \mathbf{B} \mathbf{B} \right) - Pr \nabla^2 \mathbf{u} + \nabla P + \frac{Pr}{E} \hat{z} \cdot \mathbf{u} - Pr Ra T \hat{z} = 0
\]  

(1)

\[
\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{u} \mathbf{B} - \mathbf{B} \mathbf{u}) - \frac{1}{p} \nabla^2 \mathbf{B} = 0
\]  

(2)

\[
\frac{\partial T}{\partial t} + \nabla \cdot (\mathbf{u} T) - \nabla^2 T = 0
\]  

(3)

\[
\nabla \cdot \mathbf{u} = 0
\]  

(4)

\[
\nabla \cdot \mathbf{B} = 0
\]  

(5)
where \( p \) is the pressure containing parts due to the centrifugal and Lorentz force but not the hydrostatic part. \( \mathbf{u}, \mathbf{B} \) and \( T \) are the velocity, the magnetic field and the temperature, respectively. \( \mathbf{x}_i \) is the unity vector in vertical direction. \( Pr \) is the Prandtl number defined as \( Pr = \frac{\nu}{\kappa} \) with \( \nu \) being the kinematic viscosity and \( \kappa \) the thermal diffusivity and \( p \), the Roberts number as defined by \( p = \kappa / \eta \) where \( \eta \) is the magnetic diffusivity.

For the equations, we used the Boussinesq approximation and scaled the length with the height \( d \) of the box and the time \( t \) with the thermal diffusion-scale \( d^2 / \kappa^{-1} \).

The ratio of the buoyancy forces over viscous forces is given by the Rayleigh-number

\[
Ra = \frac{\alpha g \Delta T d^3}{\kappa \nu}
\]  

(6)

where \( \alpha \) is the thermal diffusivity, \( \Delta T \) is the temperature difference over the box and \( g \) is the gravitational acceleration in \( \mathbf{x}_i \) direction.

The ratio of the viscous forces over the Coriolis forces is measured by the Ekman number, \( E \):

\[
E = \frac{\nu}{2\Omega d^2}
\]

(7)

where \( \Omega \) is the angular velocity. In the horizontal direction, we use periodic boundary conditions. The vertical boundaries are conducting perfectly the magnetic field and free-slip for the velocity. The temperatures for the upper and lower boundaries are set to the non-dimensional values 0 and 1, respectively.

### 2.2. Numerical method

The equations are solved in the primitive variable formulation on a uniform collocated grid. A collocated, instead of the commonly used staggered arrangement (Harlow and Welch, 1965) has been used since it offers more flexibility with respect to a later extension to spherical geometry. Discretization is performed by employing cell-centered finite volume discretization with a second order central scheme for the diffusive fluxes and an upwind biased scheme, which is based on a quadratic interpolation, QUICK (Leonardo, 1979). Values that are needed at cell faces are interpolated using a second order interpolation. A general description of the finite volume method is presented in Patankar (1980).

#### 2.2.1. The implicit–explicit method (IMEX)

In the implicit–explicit version, the solution has been advanced in time by employing a linear multi-step method to the Eqs. (1)–(3). Here, the convective terms are treated explicitly while diffusive terms are treated implicitly. The IMEX method (Frank et al., 1997) we have used is defined as follows.

Consider an ordinary differential equation of the form,

\[
\frac{dw}{dt}(t) = F(t,w(t)) + G(t,w(t)).
\]

(8)

Applying the IMEX method leads to the temporal discretization

\[
w^{n+1} = w^n + \frac{1}{2}(2 + \delta) \tau F^n - \frac{1}{2} \delta \tau F^{n-1}
\]

\[+ \frac{1}{2} \left( \frac{2 + \delta}{1 + \delta} \right) \tau G^{n+1} + \frac{1}{2} \left( \frac{\delta}{1 + \delta} \right) \tau G^{n-1},
\]

(9)

\[
\delta = \frac{t_{n+1} - t_n}{t_n - t_{n-1}}, \quad \tau = t_{n+1} - t_n.
\]

(10)

The IMEX scheme effectively decouples the system of equations. Recently, it has become clear that the stability of the individual implicit and explicit methods does not guarantee the stability of the combined method (Frank et al., 1997). It can, however, be shown that stability is obtained for a time step, \( \tau \), defined by:

\[
\tau = \frac{c}{\max_{i=1}^{N} \left[ \frac{\left| u_i \right| + \sqrt{Pr \left| B_i \right|}}{p} \right]}.
\]

(11)

\((u_i, B_i \text{ and } \Delta x_i \text{ are the components of the velocity, magnetic field and gridspacing, respectively with } c \text{ being smaller than 0.4 (Trompert and Hansen, 1999).})

In order to determine a divergence-free velocity field, \( \mathbf{u} \), we use a pressure correction method. In such a method, first an intermediate velocity \( \mathbf{u}^- \) is computed from known pressure values. Then a Pois-
son equation is solved for the corrected pressure, \( p^{n+1} \) and finally a new, divergence-free velocity is calculated from the corrected pressure.

We obtain the pressure correction by subtracting the equation for \( \nu^{*} \) from the one for \( u^{*+1} \) but neglecting only the viscous terms, instead of neglecting all terms with \( \nu^{*} \), as done in standard pressure correction techniques. This is done since the ratio of \( Pr/E \) can be very large and, therefore, the Coriolis terms should be included.

It is also necessary to correct the magnetic field in order to guarantee that \( \nabla B = 0 \) is fulfilled in every time step. Brackbill and Barnes (1980) have demonstrated that a nonzero \( \nabla B \) leads to physical incorrect solutions. Similarly, for the velocity correction, one first has to obtain an intermediate solution \( B^{*} \) and then can derive a Poisson equation for an auxiliary variable, \( \phi \). From \( \Delta \phi = \nabla B^{*} \), the auxiliary variable \( \phi \) can be calculated and for the corrected magnetic field \( B^{*+1} = B^{*} - \nabla \phi \), then the relation \( \nabla B^{*+1} = 0 \) will hold.

### 2.2.2. The implicit method

The results presented in this paper have been obtained by employing a fully implicit procedure to the system of equations. Time stepping is performed by a second order two-step implicit BDF (Bank, 1985) method with variable step sizes:

\[
U_{i} = a_{0} U_{i}^{n+1} + a_{1} U_{i}^{n} + a_{2} U_{i}^{n-1},
\]

with

\[
a_{0} = \frac{1 + 2 \alpha}{1 + \alpha \Delta t}, \quad a_{1} = -\frac{(1 + \alpha)^{2}}{1 + \alpha \Delta t}, \quad a_{2} = \frac{\alpha^{2}}{1 + \alpha \Delta t}, \quad \alpha = \frac{\Delta t}{\Delta t_{dd}}
\]

In the first time step, we use \( \alpha = 0 \), which results in

\[
U_{i} = \frac{1}{\Delta t} (U_{i}^{n-1} - U_{i}^{n}),
\]

which is the formula for Backward Euler (\( \alpha = 0 \)). The length of the time step is determined by the convergence of the solver. If there is no convergence after 200 iterations, the time step is reduced and the solution procedure is restarted. If the number of iteration is smaller than 10, the length of the time step is increased.

For each time step, one has to solve a nonlinear set of equations, as given by:

\[
\begin{pmatrix}
A(u) & B & C & D \\
0 & E(u) & 0 & 0 \\
0 & 0 & F(u) & 0 \\
G & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
u \\
B \\
T \\
P
\end{pmatrix}
= \begin{pmatrix} f_1 \\
0 \\
0 \\
0
\end{pmatrix}
\]

where the first row denotes the momentum equation and the second, the equation for the magnetic field, \( B \). The third row stands for the energy equation and the fourth row for the continuity equation. In order to solve the nonlinear system, we basically follow a SIMPLE strategy (Patankar, 1980). A block-symmetric Gauss–Seidel procedure is employed where blocking is applied over \( u, B \), and \( T \). As such, for every gridpoint, first a \( 7 \times 7 \) system is solved and the pressure is calculated from a Poisson equation. It is noteworthy that in the pressure correction equation, only the viscous terms are neglected, not the coriolis terms. For the solution of the pressure equation, we used a BICGSTAB procedure (van der Vorst, 1992). Subsequently, the velocity is corrected and if no convergence has been obtained, the system is further iterated. In case that no convergence can be reached after a number of iterations (usually 20), the SIMPLE procedure is repeated, employing a smaller time step. After a successful series of SIMPLE iterations on auxiliary variable, the correction of the magnetic field is calculated. Similarly, for the pressure solution procedure, a Poisson equation for an auxiliary variable can be derived. We employ BICGSTAB to solve this Poisson equation. Finally, the magnetic field is corrected, which forms the conclusion of a time step.

For parallelization, the 3D domain is decomposed into subblocks and each processor is assigned a subblock. Each subblock has a buffer zone, which is two cells wide, due to the stencil of the convection scheme. This buffer zone is used for exchanging data from other processors. A sketch of this decomposition is shown in Fig. 1. Explicit message-passing is carried out using the MPI standard. All the communication between the processors are carried out in a separate subroutine since this provides a tight synchronization between the processors. The scalability of the program is demonstrated in Fig. 2. Over a wide range, the speed increase is almost linear.
2.2.3. Model validation

The implicit–explicit version of the code has been extensively checked against various cases ranging from a vortex decay problem (Chorin, 1968) to the thermal convection problem (De Vahl and Jones, 1983). In Trompert and Hansen (1999), these tests are described in detail. In order to test the fully implicit procedure, we have conducted a comparison between the fully implicit code and the implicit–explicit version for a dynamo case. For a value of $Ra = 10^9$ and an Ekman number of $10^{-4}$, Fig. 3 shows the evolution of the Nusselt number, $Nu$ and the ME/KE ratio. Clearly, both codes produce virtually the same values, which we take as a clear indication that the fully implicit version is properly working.

3. Efficiency of the fully implicit method

Our efforts to implement a fully implicit method stem from the well known phenomenon that the velocity of the Alfvén waves increases linearly with the magnetic field. In strong field dynamos, this means a severe restriction on the time step, if explicit time stepping is employed. Fully implicit methods can potentially mean a significant advantage since they allow larger time steps, which are not restricted by the Levi–Courant criterion. On the other hand,

![Diagram](image)

Fig. 1. Sketch of the decomposition of the Cartesian domain into eight subblocks as denoted by the dashed lines. The width of the grey buffer zones is determined by the stencil of the convection scheme.

![Diagram](image)

Fig. 2. Scalability of the program for a grid resolution of $32^3$ and $64^3$. The calculations have been carried out on the 256-node T3E of the HLRZ Jülich/Germany. On the $x$-axes, we show the number of processors used, which is increased as a power of two. The program does not require the number of processors to be a power of two but one does obtain optimal results by doing so, due to the load balancing between the processors. The $y$-axes give the time needed per time step averaged over 10 time steps. We did not obtain a time for the $64^3$ for less than four processors because the problem did not fit on the local memory of the processors.
fully implicit methods are more demanding in terms of memory requirements. Moreover, it is not clear beforehand if the increased effort per time step in an implicit method ultimately means an overall gain in terms of computational efficiency. In order to answer these questions, we have compared two versions of our code: the implicit–explicit version as described in Trompert and Hansen, (1999) and a fully implicit version in which the convective terms are also treated implicitly. With regard to the memory requirements, we observe that the fully implicit code needs about three times more memory than the implicit–explicit formulation. This result has been established from both codes running on a single node of a Cray-T3E computer. For different resolutions, the results are summarized in Table 1. This finding indicates that the increased memory requirements of the fully implicit method as compared to the implicit–explicit one certainly does not mean a severe restriction. Today’s generation of parallel computer easily provides 128–512 MB local memory per node, thus, a factor of three seems not critical. In typical dynamo simulations, sufficiently long time evolutions need to be simulated in order to observe generic behavior of the dynamo. This means that subblocks of the domain, of typically not larger than $32^3$ nodes, are assigned to a single processor. In order to further speed up the calculation, one would prefer to use more processors rather than putting larger subblocks to each single node. Taking into account this aspect, we feel once more that the increased memory requirements of the implicit version is not a crucial bottleneck.

It remains to be answered if it is worth the effort to employ a fully implicit code. This question addresses a complex problem and cannot be answered easily. A very strong rotation poses a severe difficulty for the implicit–explicit method and the iterative solver often failed to converge while the fully implicit version did converge. This is only one example of a parameter range in which a comparison of the two methods simply has not been possible. In order to get an indication on the performance of the two versions, we have carried out two calculations on a grid consisting of $32^3$ nodes. A Rayleigh number of $10^5$ has been chosen while Prandtl, Ekman and Roberts numbers were kept at a value of unity. Both calculations were started from a conductive temperature profile, prescribed magnetic field and a small perturbation for the velocity field. We ran both codes for 5 min total CPU time on 32 nodes of a Cray T3E computer.

Within the allotted time 510 time steps were carried out by the implicit–explicit method while the fully implicit scheme has performed 397 steps. In terms of model time, the fully implicit scheme has clearly demonstrated its superiority. When scaled by thermal diffusion time, the implicit–explicit version, restricted by Levi–Courant criterion, reached a dimensionless time of $t = 0.055$. With the fully im-

![Fig. 3. Comparison of the development of the Nusselt number (a) and the ratio of ME/KE (b) vs. time for the implicit and the implicit–explicit scheme.](image)

<table>
<thead>
<tr>
<th>Resolution</th>
<th>Implicit–explicit (MB)</th>
<th>Fully implicit (MB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$20^3$</td>
<td>7.3</td>
<td>15</td>
</tr>
<tr>
<td>$32^3$</td>
<td>14</td>
<td>49</td>
</tr>
<tr>
<td>$50^3$</td>
<td>50</td>
<td>164</td>
</tr>
</tbody>
</table>

The resident size of the programs is shown in megabyte (MB).
Fig. 4. Size of the time step $\Delta t$ vs. time $t$ for both the implicit–explicit and the fully implicit methods. The calculations have been carried out at $Ra = 10^5$ with no rotation and show the spin up from a conductive initial state. The grid resolution is $32^3$ and both calculations have been carried out on 32 nodes on a Cray T3E with an allotted CPU time of 5 min each.

Implicit method, a model time of $t = 0.091$ has been reached. The temporal evolution of the time step is displayed in Fig. 4. Obviously, the time step limiter, which is also necessary for the full implicit code in

Fig. 5. Size of the time step vs. the number of time steps for a value of $Ra = 10^6$ and $E = 10^{-4}$. The solid line indicates the values for the fully implicit code whereas the dashed line shows the values for the implicit–explicit scheme. Starting from a conductive state, the implicit–explicit scheme initially performs larger time steps compared to the implicit scheme. After this initial spin up, the implicit scheme has larger time steps.
order to guarantee convergence of the iterative solver, is much less restrictive than the Levi–Courant criterion as applicable to the implicit–explicit method.

We have performed another test for more extreme values of the Rayleigh and Ekman numbers on a 32-node workstation cluster. During the 3000 time steps, we have monitored the size of the time step in both the implicit–explicit and the fully implicit version (Fig. 5). For $Ra = 10^9$ and $E = 10^{-5}$, we observe, in general, significantly longer time steps in the implicit procedure. However, we also observe periods in which both procedures yield to almost the same time step or the time step of the implicit–explicit method is even larger. We have not been able to pin down this problem yet. Because the communication between workstation nodes has a higher latency and a lower band width compared to the Cray T3E, the fully implicit method still perform better because less communication is needed. In general, the fully implicit method shows better convergence properties. This impression is strengthened by the observation that, as one moves further to more extreme values of parameters, the implicit–explicit method fails to converge, while convergence can easily be obtained by the fully implicit scheme. This is demonstrated in Figs. 6a,b,c which displays the evolution of the Nusselt number, the ME and the ME/KE ratio. For the parameters, chosen here to be $Ra = 10^{11}$ and $E = 10^{-5}$, we observe an exponential increase of the ME. Also, the ME/KE ratio shows an exponential growth during this period. It must be mentioned here that the calculation has been started from an initially conductive field. This is numerically challenging and may be the reason for the failure in the convergence of the implicit–explicit scheme. We think, however, that this case demon-

![Fig. 7. Spatial structure of the temperature field (yellow color) and the ME (blue color) visualized by isosurfaces. The small-scale structure of both the magnetic and the temperature fields are visible.](image)
strates the greater potential of the fully implicit method. Fig. 7 displays one snapshot of the evolution. Temperatures are shown as yellow isosurface, the distribution of ME is indicated by blue isosurfaces. Due to the strong rotation and the infinitely conducting boundary conditions, the ME distribution shows a very fine structure.

4. Dynamo simulations with varying Prandtl number

In most studies of the dynamo problem, the inertia terms have been neglected (e.g., Glatzmaier and Roberts, 1995; Kuang and Bloxham, 1997). This seems justified in the outer core since at realistically low values of the Ekman number ($E = 10^{-14}$), Coriolis forces will clearly dominate both viscous and inertial forces. In terms of dimensionless parameters, the lack of inertia is described by a value of infinity for the Prandtl number. Today’s available numerical models, however, have so far been unable to reach values of the Ekman number lower than about $10^{-5}$. The question arises if, for such values of $E$, inertia can in fact be neglected, or if a scenario with a relatively high Ekman number and an infinite Prandtl number would form a physically inconsistent model.

From the material properties of the outer core, one would estimate a value of the Prandtl number, based on the molecular viscosity, being in the interval of $0.1 - 0.01$. Assuming such values in the dynamo models would likewise lead to a dilemma since in a scenario with Prandtl between 0.1 and 0.01 and the Ekman number of $10^{-5}$, viscous forces and inertia would be much over-emphasized as compared to the Earth’s outer core. The practical approach is therefore to describe the viscosity by an eddy viscosity. Depending on the value chosen for the eddy viscosity, this results in different Prandtl numbers.

The approach we are taking here is to systematically investigate the influence of the Prandtl number on the flow. In laboratory studies, the influence of the Ekman number can quite easily be studied by varying the rotation rate, while it seems much more difficult to monitor the effect of different Prandtl numbers. Knowledge about the role of the Prandtl numbers seems also necessary in order to be able to compare the experiments that have been carried out with different working fluids (e.g., Na, Hg, Ga). We took these questions as enough reason for investigating the role of inertia on dynamo action.

In order to do so, we have carried out a series of experiments in which the Prandtl number has been systematically varied in the interval of $0.01 < Pr < 1000$. Specifically, we have chosen values for $Pr$ of 0.01, 0.1, 10 and 1000. Higher Prandtl numbers have not been considered since, first, they are not relevant for the Earth’s core and secondly, higher values of $Pr$ cause to increase the number of iterations necessary for convergence. It is important to point out that the increase in iterations with increasing stiffness of the problem (i.e., increasing $Pr$) is not a principal problem but can be overcome by altering the iteration scheme. A Childress–Soward dynamo consisting of a plane layer of electrically conducting fluid has been employed in this series of experiments. The flow is driven by a thermal gradient and rotates around the vertical axis. The upper and lower boundary are infinitely conducting and periodic boundary conditions are applied at the side walls. For simplicity a domain with aspect ratio of unity has been chosen and the values for the Rayleigh, Ekman and Robert’s number have been fixed to $Ra_s = 10^5$, $E_s = 10^{-3}$ and $p = 4$. Fig. 8 displays snapshots of the calculation. Here, the yellow isosurface for the temper-
temperature at a value of $T = 0.7$ is shown, thus displaying the position of the warm upwellings. Clearly at low Prandtl number, the flow is much more irregular while at higher values of $Pr$ the upflow takes a more regular column-like structure. At lower values of $Pr$ the isosurface encapsulates significantly more material than at higher values of $Pr$. As we have not included pressure- or temperature-dependent material properties, the flow is fully symmetric and the statements about the upwellings do likewise hold for the cold downwellings.

The blue isosurface in Fig. 8 displays the local distribution of ME. The value of the isosurface has been chosen to be 30% of the maximum ME and therefore varies for the different model calculations displayed.

We observe the ME being concentrated in small elongated filaments but we do not observe a mean magnetic field. This is due to the electrically and infinitely conducting boundaries. The results are summarized in Table 2, giving the values of the Nusselt number, the RMS velocity and the ME for each value of the Prandtl number. For the smallest Prandtl number ($Pr = 0.01$), the magnetic field dies out after a short period. For the lowest value of $Pr$, the resulting Nusselt number is also the lowest from all experiments and the RMS velocity is fluctuating between 42 and 55. Increasing the Prandtl number to 0.1 leads to a slight increase of the Nusselt number and to a significant increase of the RMS velocity. A magnetic field is clearly detectable though it is weak. Further increasing the Prandtl number to a value of 10 leads to a clear increase in Nusselt number, RMS velocity and also in the ME. From that value on, an asymptotic regime has seemingly been reached. Neither the ME nor the Nusselt number shows a further increase when the Prandtl number is increased to 1000.

5. Conclusion

We have employed a numerical model in Cartesian geometry in order to investigate various aspects of dynamo action. Our model is based on a finite volume discretization and has been designed to work efficiently on parallel computers. Thus, far, we have used an implicit–explicit time stepping scheme in which the convective terms were treated explicitly while the other terms were treated implicitly. As demonstrated, this version of the model performed well on parallel machines with respect to overall speed and with regard to scalability. In the present studies we have extended the code to a fully implicit scheme. One of the main aims of this investigation was to find out if the increased requirements of an implicit time stepping scheme with respect to both time and memory requirements, would be worth the effort. This is a complex question and it is clear that it cannot be answered for all parameter combinations, which are possibly relevant for a dynamo. For two sets of parameters, we have compared the explicit–implicit version with the fully implicit time stepping method. From this comparison, it becomes clear that the increased memory requirements of the fully implicit version forms by no means a bottleneck for today’s generation of parallel computers. Our results also indicate that the fully implicit method is superior in terms of time performance. Moreover, it seems from our experiments, that a more extreme parameter range can be reached with the implicit code while the explicit–implicit version faces convergence problems in the iterative solver. Since there is no restriction on the time step from stability considerations, a larger time step can be chosen. There is in fact another restriction on the time step stemming from convergence considerations of the iterative solver. This restriction however is demonstrated to be much less stringent than the Levi–Courant criterion. At certain parameters, within a given span of CPU-time the fully implicit version of the code enables us to simulate almost double the evolution time as compared to the implicit–explicit

<table>
<thead>
<tr>
<th>$Pr$</th>
<th>0.01</th>
<th>0.1</th>
<th>1</th>
<th>10</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Integration time ($t$)</td>
<td>10</td>
<td>12</td>
<td>4</td>
<td>0.1</td>
<td>0.5</td>
</tr>
<tr>
<td>Average Nusselt number</td>
<td>4.2</td>
<td>5.8</td>
<td>8</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>Average RMS velocity</td>
<td>42; 55</td>
<td>82</td>
<td>130</td>
<td>150</td>
<td></td>
</tr>
<tr>
<td>Average ME</td>
<td>0</td>
<td>3000</td>
<td>8000</td>
<td>8000</td>
<td></td>
</tr>
</tbody>
</table>

The averaged ME is scaled with $\sqrt{\nu \rho_0 \mu_0 \frac{1}{d}}$. 


version. This might be dependent on the actual parameter setting but we consider the result at least as an indication that a fully implicit treatment of the equations forms a superior approach to strong field dynamos. Furthermore, we have studied the influence of inertia on dynamos. In a set of experiments with fixed parameters but varying Prandtl numbers, we have shown that the structure of the flow and of the resulting magnetic field is strongly influenced. For the Childress–Soward scenario, we found that there was no magnetic field at Prandtl numbers of Pr = 0.1. With increasing Pr, we found the magnetic field energy to increase and then to reach an asymptotic value at values of Pr from around 10. These experiments leave us with the impression that the dynamo action seems to be only a little influenced by the Prandtl number in the limit of high Pr-flow. At small Pr however, the flow and also the magnetic field properties are strongly influenced by the role of inertia. Since the flow in the Earth’s outer core operates in this regime, further experiments should clarify if inertial forces need to be included when the mechanism of the geodynamo is investigated.

References